A CLASS OF NONPOSITIVELY CURVED KÄHLER MANIFOLDS BIHOLOMORPHIC TO THE UNIT BALL IN $\mathbb{C}^n$

HARISH SESHADRI AND KAUSHAL VERMA

Abstract. Let (M, g) be a simply connected complete Kähler manifold with nonpositive sectional curvature. Assume that g has constant negative holomorphic sectional curvature outside a compact set. We prove that M is then biholomorphic to the unit ball in $\mathbb{C}^n$, where $dim_{\mathbb{C}} M = n$.

Résumé. Soit (M, g) une variété kählérienne complète et simplement connexe à courbure sectionnelle non-positive. Supposons que g ait courbure sectionnelle holomorphe constante et négative en delors d'un compact. On démontre que M est biholomorphe à une boule dans $\mathbb{C}^n$, où $dim_{\mathbb{C}} M = n$.

1. Introduction

An important issue in complex differential geometry is to understand the relationship between the curvature of a Kähler manifold and the underlying complex structure. The simplest theorem along these lines is the classical theorem of Cartan that a simply connected complete Kähler of constant holomorphic sectional curvature is holomorphically isometric to $\mathbb{C}P^n$, $\mathbb{B}^n$ or $\mathbb{C}^n$ (where $\mathbb{B}^n$ is the open unit ball in $\mathbb{C}^n$) depending on whether the curvature is positive, negative or zero. Here the metrics on $\mathbb{C}P^n$, $\mathbb{B}^n$ or $\mathbb{C}^n$ are the Fubini-Study metric, the Bergman metric and the flat metric, respectively.

A far deeper theorem is that of Siu and Yau [7] which states that a complete simply connected nonpositively curved Kähler manifold of faster than quadratic curvature decay has to be biholomorphic to $\mathbb{C}^n$. An analogue of this theorem for characterizing the ball in $\mathbb{C}^n$ is not known, to the best of our knowledge. As a first step in this direction we prove the following theorem, which can also be regarded as perturbed version of Cartan's theorem stated above, at least in the negative case.

Theorem 1.1. *Let (M, g) be a simply connected complete Kähler manifold with nonpositive sectional curvature. If g has constant negative holomorphic sectional curvature outside a compact set, M is biholomorphic to the unit ball in $\mathbb{C}^n$, where $dim_{\mathbb{C}} M = n$.*

It is natural to ask if the theorem is true if we only have that the holomorphic sectional curvatures converge to -1 as $r \to \infty$, where r is the distance from a fixed point in M. However, the following class of examples show that the theorem then fails: If g is the Bergman metric of a strongly pseudoconvex domain Ω in $\mathbb{C}^n$, then the holomorphic sectional curvatures of g approach -1 as one approaches $\partial\Omega$. Moreover, by the results of [5], if Ω is a small enough perturbation of $\mathbb{B}^n$, i.e., $\partial\Omega$ is a C^∞ small perturbation of $\partial\mathbb{B}^n$, then g has negative sectional curvature. By Chern-Moser theory,"many" of these perturbations are not biholomorphic to $\mathbb{B}^n$. Hence one should impose a specified rate of convergence of holomorphic sectional

curvatures to -1 in order to obtain a theorem similar to that of Siu-Yau in the negative case. However, it is not clear what this rate of convergence should be.

Finally, we note that the hypotheses of the Siu-Yau theorem are strong enough to guarantee that the Kähler manifold is actually holomorphically *isometric* to $\mathbb{C}^n$ with the flat metric. In fact, R. Greene and H. Wu proved that a *Riemannian* manifold with the same curvature hypotheses has to be flat [6]. In our case, however, we can always perturb the Bergman metric of $\mathbb{B}^n$ on a compact set and satisfy our hypotheses.

Roughly, the proof of Theorem 1.1 proceeds as follows: Suppose that M has constant holomorphic curvature outside a compact set K. As in the proof of Cartan's theorem, one can use the exponential map to construct holomorphic maps to $\mathbb{B}^n$ on "pieces" of $M \setminus K$. The difficulty here is that even though these maps can be chosen to patch up to a give single holomorphic map from $M \setminus K$ to $\mathbb{B}^n$, this map may not be injective. We avoid this difficulty by working with ∂M, the asymptotic boundary of M. More precisely, we use the holomorphic maps above to define a spherical CR-structure on ∂M. Since ∂M is simply connected, one gets a global CR-diffeomorphism to S^{2n-1}. One then notes that since M is Stein, we can extend this diffeomorphism to M by Hartogs' theorem.

2. Proof

For the rest of this paper, M will denote a simply-connected, complete Kähler manifold with nonpositive sectional curvature and constant holomorphic sectional curvature -1 outside a compact set. ∂M will denote its asymptotic boundary. There is a natural topology, described in the proof below, on $\overline{M} := M \cup \partial M$ which makes it a compact topological manifold-with-boundary.

The main theorem is proved by first proving the following proposition. In what follows S^{2n-1} is the unit sphere in $\mathbb{C}^n$ with the induced CR-structure.

Lemma 2.1. *$\overline{M}$ can be given the structure of a smooth compact manifold-with-boundary such that ∂M admits a "natural" CR-structure which makes it CR-diffeomorphic to S^{2n-1}.*

Before beginning the proof, we recall certain general constructions on nonpositively curved manifolds:

First, we define the *"modified"* exponential map. Let V be a complex vector space with a Hermitian inner product h and let $B = \{x \in V : \|x\| < 1\}$ denote the open unit ball in V. Define the homeomorphism $\phi : V \to B$ by $\phi(x) = (1 - e^{-\|x\|})\,\frac{x}{\|x\|}$. Note that ϕ is a diffeomorphism on $V \setminus \{0\}$. When $V = T_pM$ and $h = g_p$, we will use the notation ϕ_p. For any $p \in M$, define the *modified exponential map* $\widetilde{exp}_p : B_p \to M$ by $\widetilde{exp}_p = exp_p \circ \phi_p^{-1}$.

Next, let $\partial M = \{$ equivalence classes of geodesic rays in $M\}$, where geodesics $c_1, c_2 : [0, \infty) \to M$ are equivalent if there is a constant $a < \infty$ such that $d(c_1(t), c_2(t)) < a$ for all $t \geq 0$. ∂M is usually referred to as the *asymptotic boundary* of M. We endow $\overline{M} = M \cup \partial M$ with the "cone" topology. This is the topology generated by open sets in M and "cones", corresponding to $x \in M$, $z \in \partial M$ and $\varepsilon > 0$, defined by

$$C_x(z, \varepsilon) := \{y \in \overline{M} \mid y \neq x \;\text{ and }\; <_x (z, y) < \varepsilon\}.$$

Here the angle $<_x (z,y) := <(c_1'(0), c_2'(0))$ where c_1 and c_2 are geodesics joining x with z and y (see [1] for details). For any $p \in M$, $\widetilde{exp}_p$ extends to a homeomorphism, which we continue to denote by the same symbol, from $\overline{B}_p$ to $\overline{M}$.

Now we come to the proof of Lemma 2.1.

Proof. Suppose M has constant holomorphic curvature -1 outside a compact set K. Fix $o \in M$. Choose R large so that $d(o,x) < R$ for any $x \in K$.

If $p \in \partial M$, then there is an unit-speed geodesic $\gamma_p : [0,\infty) \to \overline{M}$ with $\gamma_p(0) = o$ and $lim_{t\to\infty}\gamma_p(t) = p$. Let

$$x(p) := \gamma_p(R).$$

We observe that $C_{x(p)}(p, \frac{\pi}{4}) \cap K = \emptyset$. This is because $d(o,x) > R$ for any $x \in C_{x(p)}(p, \frac{\pi}{4})$ by Toponogov's Comparison Theorem for geodesic triangles in nonpositively curved manifolds (see [1], Page 5). Hence, by our choice of R, g has constant holomorphic sectional curvature in the interior of $C_{x(p)}(p, \frac{\pi}{4})$.

Choose $p_1, .., p_k \in \partial M$ so that if

$$U_i := C_{x(p_i)}(p_i, \frac{\pi}{4}),$$

then $U_1 \cap \partial M, .., U_k \cap \partial M$ cover ∂M.

For $i = 1, .., k$, choose linear isometries $L_i : T_{\gamma_{p_i}(R)} \to T_0\mathbb{B}^n$. We then get maps

$$f_i := \widetilde{exp}_0 \circ L_i \circ \widetilde{exp}^{-1}_{x(p_i)} : U_i \to \overline{\mathbb{B}^n}.$$

These maps are homeomorphisms onto their images and we declare these to be charts on $\partial M \subset \overline{M}$. In order to check that the transition functions are smooth, let us observe the following:

First, it is easily checked that $f_i|_{U_i} = exp_0 \circ L_i \circ exp^{-1}_{p_i}$. Recall that our metric is locally symmetric in the interior of U_i. Hence by the Cartan-Ambrose-Hicks Theorem (cf. [4]), f_i is a holomorphic local isometry there.

Next, Toponogov's Comparison Theorem implies that for any $p \in \partial M$, U_p is geodesically convex. Also, it is clear from the definition that if $q \in U_p$, then the geodesic ray starting at $x(p)$ and passing through q lies in U_q. Combining these observations, we see that if $U_i \cap U_j \neq \emptyset$, then

$$U_i \cap U_j \;\; is \;\; connected \;\; and \;\; U_i \cap U_j \cap \partial M \neq \emptyset. \tag{2.1}$$

Now the transition function $f_i \circ f_j^{-1}$ is a holomorphic isometry (for the restriction of the Bergman metric of $\mathbb{B}^n$) from $f_j(U_i \cap U_j) \cap \mathbb{B}^n$ to $f_i(U_i \cap U_j) \cap \mathbb{B}^n$. Since $f_j(U_i \cap U_j)$ is connected by (2.1), such a mapping has to be the restriction of a global automorphism of $\mathbb{B}^n$. In particular, the mapping is smooth up to the boundary, i.e. $f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ is smooth. This gives us the required smooth structure on $\overline{M}$.

Also, it is clear that the charts $(U_i \cap \partial M, f_i)$ define a CR-structure on ∂M, since the transition functions will be local CR-diffeomorphisms of S^{2n-1}. Moreover, by definition this CR-structure is locally spherical. Since ∂M is compact and simply connected, the results of [2] (basically a developing map argument) imply that there is a global diffeomorphism ψ from ∂M to S^{2n-1}. This proves the lemma. □

We continue with the proof of the main theorem, using the notation in the proof of the lemma. Let us note that by composing with holomorphic automorphisms of $\mathbb{B}^n$, if necessary, we can assume that

$$f_i|_{U_i \cap \partial M} = \psi. \tag{2.2}$$

By (2.1), (2.2) and unique continuation, if $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1} = id : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$, since $f_i \circ f_j^{-1} = id$ on $f_j(U_i \cap U_j) \cap \partial \mathbb{B}^n$.

Hence if $U_i \cap U_j \neq \emptyset$, then $f_i = f_j$ on $U_i \cap U_j$. Therefore, the f_i patch up to give a smooth mapping $F : U \to \mathbb{B}^n$ on the open neighbourhood $U = U_1 \cup ... \cup U_k$ of ∂M and F is holomorphic on $U \cap M$. Since $F|_{\partial M} = \psi$ is injective and since F is a local diffeomorphism, we can choose a neighbourhood $V \subset U$ of ∂M such that $F|_V$ is injective.

To extend F to M, we recall Wu's theorem [8] that *a simply connected complete Kähler manifold of nonpositive sectional curvature is Stein.* Combining this with the fact that $M \setminus U$ is compact, we conclude that F extends to all of M by Hartogs' theorem on Stein manifolds. By the maximum principle, $F(M) \subset \mathbb{B}^n$.

To construct the inverse of F, let $G = F|_V^{-1}$. G is smooth map defined on the neighbourhood $F(V)$ of $\partial \mathbb{B}^n$, which is holomorphic in $F(V) \cap \mathbb{B}^n$. Since M is Stein, M is an embedded submanifold of some $\mathbb{C}^N$. Again by Hartogs theorem and the maximum principle, $G : F(V) \cap \mathbb{B}^n \to V \subset \mathbb{C}^N$ extends to a smooth map $G : \overline{\mathbb{B}^n} \to \overline{M}$ which is holomorphic in $\mathbb{B}^n$.

Finally, by unique continuation, $F \circ G = id_{\overline{\mathbb{B}^n}}$ and $G \circ F = id_{\overline{M}}$ Q.E.D.

Remark: A CR-structure on the boundary of a nonpositively curved Kähler manifold is shown to exist under hypotheses more general than ours in [3].

We end with the following

Question: Let (M^n, g) be a simply-connected complete Kähler manifold of nonpositive curvature. If g is locally symmetric outside a compact set, is M biholomorphic to $\Omega \times \mathbb{C}^{n-k}$, where Ω is a bounded symmetric domain in $\mathbb{C}^k$, for some k ?

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Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: harish@math.iisc.ernet.in

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: harish@math.iisc.ernet.in